oscillations is the torquing of the INS platform. Using tangent plane coordinates, it was then shown that the torquing was not the reason. We then explained that Schuler oscillations stem from two factors acting together. One is the fact that accelerometers measure specific force rather than acceleration and therefore the gravity part of the accelerometer reading needs to be removed, thereby creating a feedback loop. The second factor is the gravity field being a central field. Although the explanations given here utilized stable platforms, the same can be shown for strapdown INSs.

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Controllability and Optimization in Aeroassisted Orbital Transfer

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I. Introduction

THIS Note addresses the atmospheric flight phase of an aeroassisted orbital transfer maneuver of a spacecraft. The re-entry mission of such a spacecraft has been studied recently by Roenneke and Cornwell.[1] The primary objective is to relate the computation of optimal trajectories to the controllability properties of the spacecraft. The vehicle is assumed to be controlled by the bank angle and is to perform a zero orbital plane change with some fixed exit conditions. A performance index is chosen such that the exit velocity error and the exit wedge angle will be minimized.

Commonly, employed optimization procedures are based on the iterated shooting method.[2] Such procedures are known to be highly sensitive to the initialization of the adjoint vector. Finding an initial condition of the adjoint vector that ensures the convergence of the optimization procedure is often problematic. Indeed, the initial adjoint vector must be chosen close to the optimum value, which is of course not known a priori. The goal of this Note is not to contribute to optimization theory but rather to highlight the controllability aspects that render the optimization problem ill conditioned. This will enable the systematic use of optimization for generating trajectories in those areas where the optimization problem is well posed. Indeed, in the higher layers of the atmosphere, the atmospheric density is small and the spacecraft's controllability vanishes. Simulations show that the optimal-control method is more efficient when applied in the lower layers of the atmosphere.

The Note is organized as follows. In Sec. II, the equations of motion of the vehicle are described, and in Sec. III the optimal-control method is outlined. The controllability analysis is presented in Sec. IV, and finally, the simulation results are given in Sec. V.

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II. Model Equations

The basic equations for orbital transfer are those for deorbit, atmospheric flight, boost, and circularization or reorbit. For guidance purpose, however, we consider only the atmospheric flight portion of the maneuver. The aerodynamic forces are limited to *drag* and *lift*, and the vehicle is assumed to be axisymmetric. The equations of motion of the center of mass are given by

$$\dot{r} = V \sin\gamma$$

$$\dot{\tau} = V \cos\gamma \cos\chi / (r \cos\delta)$$

$$\dot{\delta} = V \cos\gamma \sin\chi / r$$

$$\begin{aligned}
\dot{\chi} &= -\frac{\boldsymbol{L}\sin\sigma}{mV\cos\gamma} - \frac{V}{r}\cos\gamma\cos\chi\tan\delta - 2\omega(\sin\delta \\
&\quad - \tan\gamma\cos\delta\sin\chi) - \frac{\omega^2 r}{V\cos\gamma}\sin\delta\cos\delta\cos\chi \\
\dot{V} &= \frac{\boldsymbol{D}}{m} - \frac{\mu}{r^2}\sin\gamma + \omega^2 r\cos\delta(\sin\gamma\cos\delta \\
&\quad - \cos\gamma\sin\delta\sin\chi) \\
\dot{\gamma} &= -\frac{\boldsymbol{L}\cos\sigma}{mV} + \left(\frac{V}{r} - \frac{\mu}{r^2 V}\right)\cos\gamma + 2\omega\cos\delta\cos\chi \\
&\quad + \frac{\omega^2 r}{V}\cos\delta(\cos\gamma\cos\delta + \sin\gamma\sin\delta\sin\chi)
\end{aligned} \tag{1}$$

where $\boldsymbol{D}$ and $\boldsymbol{L}$ are the drag and lift forces, r is the distance from the center of Earth, δ is the latitude, τ is the longitude, V is the velocity, γ is the flight path angle, χ is the heading angle, m is the vehicle's mass (=1408.6 kg), C_D and C_L are the drag and lift coefficients, σ is the bank angle, ω is the angular velocity of Earth ($=0.729\ 10^{-4}$ rad/s), and μ is the gravitational constant of Earth ($=0.3986\ 10^{15}$ m^3/s^2). The simulation results presented later use the 1976 U.S. Standard Atmosphere.

III. Optimal Control

The objective is to bring the spacecraft from a high orbit to a low orbit using the natural braking action of the atmosphere. Certain atmospheric exit conditions have to be fulfilled so that the vehicle reaches the low orbit with a suitable velocity. These conditions consist of a given atmospheric exit velocity and zero exit wedge angle η. In order to match the final conditions, the following performance index is minimized:

$$\mathcal{J} = g_1(\Delta h)^2 + g_2(\Delta V)^2 + g_3(\eta)^2 \tag{2}$$

where g_1, g_2, g_3 are positive, Δh represents the final altitude error, ΔV is the atmospheric exit velocity error, and η is the final wedge angle.

By using the principle of conservation of energy and angular momentum at the atmospheric exit and the point of the low orbit, the relation between the exit velocity ($\tilde{V}_f$) and the exit flight path angle ($\tilde{\gamma}_f$) computed in an inertial system is derived. This relation is expressed by[3]

$$r_l^2\left(2V_{at} - \tilde{V}_f^2\right) - 2r_l r_a V_{at}^2 + r_a^2 \tilde{V}_f^2 \cos^2\tilde{\gamma}_f = 0 \tag{3}$$

where r_a is the radius of atmospheric boundary, V_{at} is the circular velocity at $r = r_a$, and r_l is the radius of a low orbit to reach in the next phase of the maneuver. Equation (3) guarantees that, after exiting, the spacecraft ascends to the desired low orbit. Thus the atmospheric exit velocity error is given by

$$\Delta V = V_{at}\sqrt{\frac{\left[2r_l^2/r_a(r_a-1)\right](V^2+\alpha)}{V^2\left(r_l^2/r_a^2-\cos^2\gamma\right)+\alpha\left(r_l^2/r_a^2-1\right)}}-\sqrt{V^2+\alpha} \tag{4}$$

where $\alpha = 2\omega r V\cos\gamma\cos\chi\cos\delta + \omega^2 r^2\cos^2\delta$.

The optimization procedure is simplified by assuming that ΔV is the difference between the desired exit velocity (V_{de}) and the real exit velocity. This is justified by the fact that once an optimum value V_f is obtained, Eq. (3) is verified. Another simplification will be made by taking $g_3 = 0$. In this case, the term corresponding to η in $\mathcal{J}$ is neglected. This simplification is justified by simulations. If h_{de} is the desired altitude, the performance index then becomes

$$\mathcal{J} = g_1(h-h_{de})^2 + g_2(V-V_{de})^2 \tag{5}$$

The first step in the optimization procedure is to formulate the Hamiltonian as

$$\mathcal{H} = \lambda_1\dot{r} + \lambda_2\dot{\tau} + \lambda_3\dot{\delta} + \lambda_4\dot{\chi} + \lambda_5\dot{V} + \lambda_6\dot{\gamma} \tag{6}$$

where $\lambda_i, i = 1,\dots,6$, are the costates and $\lambda = (\lambda_1,\dots,\lambda_6)^T$ is the adjoint vector. The initial time t_0 is fixed at $t_0 = 0$ and the final time t_f is free and is to be determined as the time at which the spacecraft exits the atmosphere.

The equation for the optimal control of the bank angle is given by $\partial\mathcal{H}/\partial u = 0$. It yields

$$u^* = \arctan\frac{\lambda_4}{\lambda_6\cos\gamma} \tag{7}$$

and the costate (adjoint) equations are given by

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial\mathcal{H}}{\partial r}, & \dot{\lambda}_2 &= -\frac{\partial\mathcal{H}}{\partial\tau}, & \dot{\lambda}_3 &= -\frac{\partial\mathcal{H}}{\partial\delta}\\ \dot{\lambda}_4 &= -\frac{\partial\mathcal{H}}{\partial\chi}, & \dot{\lambda}_5 &= -\frac{\partial\mathcal{H}}{\partial V}, & \dot{\lambda}_6 &= -\frac{\partial\mathcal{H}}{\partial\gamma}\end{aligned} \tag{8}$$

The initial states and t_0 are assumed to be known, and the transversality conditions on the costates give the boundary conditions as follows:

$$\begin{aligned}\lambda_1(t_f) &= 2g_1(h(t_f)-h_{de})\\ \lambda_5(t_f) &= 2g_2(V(t_f)-V_{de})\end{aligned} \tag{9}$$

The system (1) is nonlinear, and thus its solution must be obtained by a numerical integration method, such as Runge–Kutta. Moreover, the problem is not a classical two-point boundary-value problem, since the final conditions are not known entirely (neither the states nor the costates); however they must satisfy the equalities (9).

The method proposed here is to solve the problem iteratively in the initial adjoint vector condition $\lambda(t_0)$ by minimizing an error criterion of the form

$$\begin{aligned}\mathcal{E}(t_f) &= k_1[\lambda_1(t_f) - 2g_1(h(t_f)-h_{de})]^2\\ &\quad + k_2[\lambda_5(t_f) - 2g_2(V(t_f)-V_{de})]^2\end{aligned} \tag{10}$$

The computation stops when $\mathcal{E}(t_f) \le \epsilon$ is satisfied, where ϵ is positive and represents a given desired accuracy for the final conditions. To start the procedure, we give the system's parameters ϵ, t_0 and initial states. Then $\lambda(t_0)$ is chosen, and we determine by numerical integration the solution of the system and the adjoint system. The error criterion $\mathcal{E}(t_f)$ is computed and compared to ϵ. If $\mathcal{E}(t_f) > \epsilon$, one modifies $\lambda(t_0)$ and resumes the process.

The minimization of $\mathcal{E}(t_f)$ with respect to $\lambda(t_0)$ can be achieved by using a POWELL-type optimization algorithm.[4] When the solution of the system and the adjoint system satisfy $\mathcal{E}(t_f) \le \epsilon$, we consider that it is optimal and we determine the optimal control u^*.

By using this POWELL algorithm, a value of $\lambda^*(t_0)$ can be obtained that satisfies the final conditions, but the convergence of the algorithm depends strongly on the value of $\lambda(t_0)$ given in the first step. Here, $\lambda(t_0)$ has to be chosen close to the optimum value in order to ensure convergence. This is due to the ill-conditioning of the problem (lack of controllability). To resolve this difficulty, a controllability analysis is carried out in the following section. It is shown that the optimization procedure is improved when taking into account the controllability of the system: the initialization of the adjoint vector becomes less crucial.

IV. Controllability Analysis

The purpose of this analysis is to ensure the convergence of the optimization routine (10) for $\lambda(t_0)$.

A controllability analysis[5] was made along the reference trajectory by introducing the so-called accessibility cospaces, which play a role dual to that of controllability distributions. For this, we consider a nonlinear system given by $\dot{x} = f(x,u)$, where $x \in R^n$ represents the states and $u \in R^m$ represents the inputs. In our case, we set $x = (r,\tau,\delta,\chi,V,\gamma)^T$ and $u = \sigma$. Let $\mathcal{X} = \text{span}\{dx\}$ where dx is the differential of x, $\mathcal{U} = \text{span}\{du\}$ where du is the differential of u, and $\dot{\mathcal{X}}$ is the time derivative of $\mathcal{X}$ along the trajectory of the system. Thus, one has

$$\dot{\mathcal{X}} = \text{span}\left\{\frac{\partial f(x,u)}{\partial x}dx + \frac{\partial f(x,u)}{\partial u}du\right\} \tag{11}$$

where $\partial f(x,u)/\partial x$ is a square matrix (6×6) and $\partial f(x,u)/\partial u$ is a vector. The special case where $\partial f(x,u)/\partial u = 0$ corresponds to an autonomous system (a system without inputs), i.e., $\dot{\mathcal{X}} \subset \mathcal{X}$.

A measure of the system's controllability is made by evaluating θ as $\|\partial f(x,u)/\partial u\|$. This controllability analysis approach is applied at different positions of the spacecraft along the reference trajectory. The simulations gave us three important zones of the altitude, corresponding to different θ. The first zone is comprised between 120 and 100 km, and the corresponding θ is 10^{-5}. The second is from 100 to 90 km, and the corresponding θ equals 10^{-4}. Finally, the third zone is the area for which the altitude is lower than 90 km, and $\theta = 10^{-3}$ is obtained. Clearly, the value of θ increases when we are in the lower layer. This means that the system is more responsive to control actions. So, one concludes that, in the lowest layer, the controllability along the reference trajectory is 100 times "greater" than in the highest layer of the atmosphere.

These results are consistent with the test of convergence of the optimization algorithm with respect to the initialization of the adjoint vector $\lambda(t_0)$. Indeed, the admissible domain of $\lambda(t_0)$ ensuring the convergence is wider when θ increases.

We carry out the study in three cases. In the first case, we apply the optimal control for full atmospheric flight ($h_e := h_f := 121$ km). In the second case, we start and stop at a lower altitude ($h_e := h_f := 100$ km), and in the last case the optimal control is only applied in the lowest layer ($h_e := h_f := 90$ km). In each of the three cases, we obtain an initial adjoint vector value $\lambda^*(t_0)$ corresponding to the optimal trajectory. This value is called the nominal value. Some variations are given to the latter and the convergence of the algorithm is observed. It is seen that the initialization of λ_1 and λ_5 is not sensitive to controllability in a very significant way. For the other variables of adjoint vector, their acceptable variations ensuring the convergence of the solution increase when we are at a lower altitude. In the first case, $\lambda_2^*(t_0)$ can vary from -10 to $+1\%$, $\lambda_3^*(t_0)$ takes a domain of variation from -50 to $+10\%$, whereas $\lambda_4^*(t_0)$ can vary up to $+10\%$ and the acceptable variation of $\lambda_6^*(t_0)$ is from -10 to $+5\%$. In the second case, the admissible domain of $\lambda^*(t_0)$ is extended, and in the third case this domain is even wider: $\lambda_2^*(t_0)$ is varying from -30 to $+10\%$, $\lambda_3^*(t_0)$ can vary from -100 to $+100\%$, the variation of $\lambda_4^*(t_0)$ is from -10 to $+30\%$, and finally, $\lambda_6^*(t_0)$ is varying from -5 to $+30\%$.

According to the above results, the acceptable domain of the initial adjoint vector value increases when the optimal control is applied at a lower altitude. This is due to the lack of controllability in the higher layers of the atmosphere. It is thus seen to be more efficient to apply the optimal control only in the section of altitude where the system is more controllable. This is the only situation where the guidance problem is significant.

Another approach analyzing the system's controllability is given,[6] where the observability analysis is also discussed. Based on the input

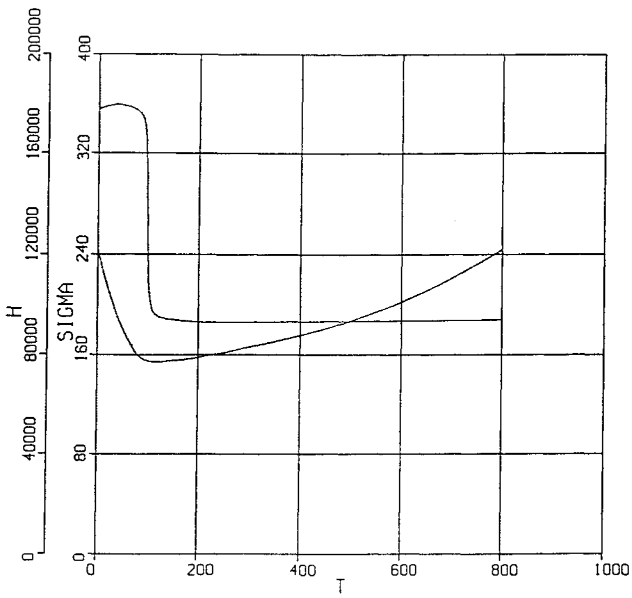

Fig. 1 Altitude h and bank angle σ vs time (full atmospheric flight).

energy, the controllability function is defined. By a state transformation, the considered system can be brought into a balanced form. In this form, the controllability function displays singular-value functions. If a singular-value function is small, then the corresponding state component is weakly controllable.

V. Numerical Results

The data used in the numerical experiments presented here are

$$g_1 = 10^{-4}, g_2 = 10^{-5}, \text{ and } k_1 = k_2 = 1$$

Full atmospheric flight trajectory. Initial conditions (entry into the atmosphere) are $h_e = 121.896$ km, $\delta_e = -4.487$ deg, $\tau_e = -134.519$ deg, $\chi_e = 29.422$ deg, $V_e = 9894.121$ m/s, and $\gamma_e = -4.674$ deg. Desired final conditions (exit from the atmosphere) are $h_{de} = 122.139$ km and $V_{de} = 7615.411$ m/s.

Using the above data, simulations were carried out. To achieve the final conditions, the POWELL algorithm gives $\lambda_1^*(t_0) = 8.519 \times 10^{-8}$, $\lambda_2^*(t_0) = 0.999$, $\lambda_3^*(t_0) = -9.257 \times 10^{-3}$, $\lambda_4^*(t_0) = -0.1695$, $\lambda_5^*(t_0) = -2.658 \times 10^{-5}$, and $\lambda_6^*(t_0) = 0.887$. The final conditions are met, and this is shown by $\mathcal{E}(t_f) = 3.925 \times 10^{-3}$. The total flight duration is 800 s. We obtain the exit velocity $V_f = 7469.840$ m/s at the altitude $h_f = 122.126$ km and the wedge angle $\eta = 1.180$ deg. Figure 1 shows the time history of altitude and control. The spacecraft enters the atmosphere at altitude $h_e = 121.896$ km and exits at $h_f = 122.126$ km. The minimum altitude reached is 76.678 km. When the vehicle enters the atmosphere, the lift is maximally upward. In this phase, the bank angle is close to 360 deg, whereas in the atmospheric exit phase, the bank angle is close to 180 deg. These simulation results are similar to those obtained by Miele et al.[3,7]

Reduced atmospheric flight trajectory. Initial conditions are $h_e = 100.195$ km, $\delta_e = -3.155$ deg, $\tau_e = -132.161$ deg, $\chi_e = 29.587$ deg, $V_e = 9912.223$ m/s, and $\gamma_e = -3.428$ deg. Desired final conditions are $h_{de} = 100.285$ km and $V_{de} = 7509.270$ m/s.

In this case, the components of initial adjoint vector given by the POWELL algorithm are $\lambda_1^*(t_0) = 2.799 \times 10^{-7}$, $\lambda_2^*(t_0) = 2.278$, $\lambda_3^*(t_0) = -8.641 \times 10^{-3}$, $\lambda_4^*(t_0) = -5.447 \times 10^{-2}$, $\lambda_5^*(t_0) = -2.655 \times 10^{-5}$, and $\lambda_6^*(t_0) = 0.881$. By applying these values, the final conditions are reached, and $\mathcal{E}(t_f) = 1.930 \times 10^{-3}$ is obtained.

Using online computation, once a reference trajectory is determined by the optimization, it is possible to solve the trajectory tracking problem.[8,9]

VI. Conclusions

The optimal-control for the atmospheric flight phase of a spacecraft is treated. Due to the sensitivity of the optimization routine to the initialization of the adjoint vector, one has to choose the latter close to the optimum value. This is not compatible with the goods of the user.

By analyzing the vehicle's controllability, it is shown that the acceptable domain of initialization for the adjoint vector ensuring the convergence of the solution increases when the flight is considered in the lower altitude, i.e., when the controllability increases. The controllability analysis provides the domain where the optimization makes sense. Therefore, the optimal control should be applied to determine the reference trajectory only in the lower layers of the atmosphere. In the higher layers, the trajectory is not responsive to the input anyway.

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Shaping Time Response by State Feedback in Minimum-Phase Systems

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Introduction

IT is well known that for linear time-invariant multiple input–output state variable systems, controllability is a necessary and sufficient condition for eigenvalue assignment (pole placement) by means of state feedback.[1–3] For multivariable systems, the feedback gains are nonunique. This nonuniqueness has been utilized for shaping the response of the system by assigning selected eigenvectors.[4,5] On the other hand, for single-input systems, once the location of the closed-loop poles is specified, the feedback gains are unique. There is no additional freedom for shaping the response of the system. In this Note, we propose a systematic approach for shaping the time response of single-input, single-output systems.

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